Math 10B with Professor Stankova
Worksheet, Discussion #26; Tuesday, 4/30/2019
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Test for Independence

Concepts

1. To test for independence, it is just a modified version of the χ^2 test. You sum up the rows to get N_i and the columns to get M_j. Let the total sum of all the elements be S. Then, your expected distribution at square ij is $\frac{N_i M_j}{S}$, and then you perform the χ^2 test. If you have r rows and c columns, then the number of degrees of freedom is $(r-1)(c-1)$.

Examples

2. The following are the actual exit poll results from the 2016 election. Is who you vote for and your age independent?

	18-24	25-29	30-39	40-49	50-64	≥ 65
Clinton	1375	1194	2129	2146	3242	1768
Trump	835	840	1628	2286	3831	2043
Other	246	177	418	233	295	118

Solution: We fill out the table with the sums to get:

	18-24	25-29	30-39	40-49	50-64	≥ 65	
Clinton	1375	1194	2129	2146	3242	1768	11854
Trump	835	840	1628	2286	3831	2043	11463
Other	246	177	418	233	295	118	1487
	2456	2211	4175	4665	7368	3929	24804

Now we can create an expected value table. If the values were independent, then for instance, the percentage of 30-39 year olds who support Clinton should be the percentage of Clinton supporters times the percentage of $30-39$ year olds or $\frac{11854}{24804} \cdot \frac{4175}{24804}$. Filling out the table with this data gives the following values:

1173.739074	1056.651911	1995.260845	2229.435172	3521.217223	1877.695775
1135.023706	1021.798621	1929.447871	2155.898041	3405.071118	1815.760643
147.2372198	132.5494678	250.2912837	279.6667876	441.7116594	235.5435817

And computing the statistic gives: 650.0178363. The critical value for $(6-1)(3-1) = 10$ degrees of freedom is 18.307. Thus, we can reject the null hypothesis and say that these two are related.

Problems

3. True **FALSE** In the homework problem, there is a contradiction because we both keep and reject the null hypothesis that the Mendelian plants are distributed $9:3:3:1$.

Solution: We reject the independence test and keep the normal χ^2 one. One explanation for this is that the plant traits are distributed that way but are not independent (i.e. sex-linked traits).

4. You are wondering whether performing well in this course and gender are related and you get the following table. Are they related?

	Male	Female
Pass	175	725
Fail	25	75

Solution: There are a total of 900/1000 people who pass and 100/1000 who fail, and 200/1000 who are male and 800/1000 who are female. Thus, if they were independent, for instance we would expect that $\frac{800}{1000} \cdot \frac{900}{1000} = 72\%$ of people to be female and pass. We can fill out the expected table as follows:

	Male	Female
Pass	180	720
Fail	20	80

Now we can do the χ^2 test to get a statistic of

$$\frac{(175-180)^2}{180} + \frac{(725-720)^2}{720} + \frac{(25-20)^2}{20} + \frac{(75-80)^2}{80} = 1.7.$$

The critical value for 1 degree of freedom is 3.841 and $1.7 < 3.841$ so we cannot reject the null hypothesis.

5. You are wondering whether performing well in this course and gender are related and you get the following table. Are they related?

	Male	Female
Pass	315	485
Fail	85	115

Solution: There are a total of 800/1000 people who pass and 200/1000 who fail, and 400/1000 who are male and 600/1000 who are female. Thus, if they were independent, for instance we would expect that $\frac{800}{1000} \cdot \frac{600}{1000} = 48\%$ of people to be female and pass. We can fill out the expected table as follows:

	Male	Female
Pass	320	480
Fail	80	120

Now we can do the χ^2 test to get a statistic of

$$\frac{(315-320)^2}{320}+\frac{(485-480)^2}{480}+\frac{(85-80)^2}{80}+\frac{(115-120)^2}{120}=0.651.$$

The critical value for 1 degree of freedom is 3.841 and $0.651 < 3.841$ so we cannot reject the null hypothesis.

6. Prove that the estimator for p of a geometric distribution is biased and an overestimate for sample size $n = 1$.

Solution: We set $\hat{p} = \frac{1}{\hat{\mu}+1} = \frac{1}{X+1}$. Then

$$\begin{aligned} E[\hat{p}] = E[1/(X+1)] &= \frac{1}{0+1}P(X=0)+\frac{1}{1+1}P(X=1)+\cdots \\ &= p+\frac{1}{2}(1-p)p+\frac{1}{3}(1-p)^2p+\cdots \\ &> p. \end{aligned}$$

Miscellaneous

Examples

7. Let $v = (1, 2, 2, -1)$ and $w = (5, 3, -5, 3)$. Calculate $v \bullet w$ and $|v|$.

Solution: $v \bullet w = 1\cdot 5+2\cdot 3+2\cdot(-5)+(-1)\cdot 3 = -2$. $|v| = \sqrt{1^2+2^2+2^2+(-1)^2} = \sqrt{10}$.

8. Calculate the partial derivatives of $f = 7 - x^2y^3$.

Solution: $f_x = -2xy^3, f_y = -3x^2y^2$. $f_{xx} = -2y^3$, $f_{yx} = f_{xy} = -6xy^2$, $f_{yy} = -6x^2y$.

Problems

9. Find the angle between the two vector $v = (1, 3, 5, -2, 4, 3)$ and $w = (1, 1, 5, 2, 2, 1)$.

Solution: If θ is the angle between them, then

$$\cos\theta = \frac{v \bullet w}{|v| \cdot |w|} = \frac{36}{\sqrt{64} \cdot \sqrt{36}} = \frac{36}{8 \cdot 6} = \frac{3}{4}.$$

Thus $\theta = \arccos(3/4) \approx 0.7227$.

10. When is $|\vec{v} \bullet \vec{w}| = |\vec{v}| \cdot |\vec{w}|$? (Hint: What is θ?)

Solution: We know that $|\vec{v} \bullet \vec{w}| = ||v| \cdot |w| \cdot \cos\theta| = |v| \cdot |w| \cdot |\cos\theta|$. Thus $|\cos\theta| = 1$ and hence $\alpha = 0, \pi$. Therefore, the vectors must on the same line.

11. Let $u = x^5y^4 - 3x^2y^3 + 2x^2$. Calculate u_{xx}, u_{xy}, u_{yx}, and u_{yy}.

Solution: $u_x = 5x^4y^4 - 6xy^3 + 4x, u_y = 4x^5y^3 - 9x^2y^2$.

$$\begin{aligned} u_{xx} &= 20x^3y^4 - 6y^3 + 4 \\ u_{xy} = u_{yx} &= 20x^4y^3 - 18xy^2 \\ u_{yy} &= 12x^5y^2 - 18x^2y \end{aligned}$$

Chi-square Distribution Table

d.f.	.995	.99	.975	.95	.9	.1	.05	.025	.01
1	0.00	0.00	0.00	0.00	0.02	2.71	3.84	5.02	6.63
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69
14	4.07	4.66	5.63	6.57	7.79	21.06	23.68	26.12	29.14
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
32	15.13	16.36	18.29	20.07	22.27	42.58	46.19	49.48	53.49
34	16.50	17.79	19.81	21.66	23.95	44.90	48.60	51.97	56.06
38	19.29	20.69	22.88	24.88	27.34	49.51	53.38	56.90	61.16
42	22.14	23.65	26.00	28.14	30.77	54.09	58.12	61.78	66.21
46	25.04	26.66	29.16	31.44	34.22	58.64	62.83	66.62	71.20
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
55	31.73	33.57	36.40	38.96	42.06	68.80	73.31	77.38	82.29
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
65	39.38	41.44	44.60	47.45	50.88	79.97	84.82	89.18	94.42
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.43
75	47.21	49.48	52.94	56.05	59.79	91.06	96.22	100.84	106.39
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33
85	55.17	57.63	61.39	64.75	68.78	102.08	107.52	112.39	118.24
90	59.20	61.75	65.65	69.13	73.29	107.57	113.15	118.14	124.12
95	63.25	65.90	69.92	73.52	77.82	113.04	118.75	123.86	129.97
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81